Kinetic Equations

Text of the Exercises

– 29.04.2021 –

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Exercise 1

Let $f : \mathbb{R}^3 \to \mathbb{R}_+$ be a regular enough function, decaying sufficiently fast at infinity. Prove that the following statements are equivalent:

(i) $\displaystyle\int_{\mathbb{R}^3} Q(f,f) \log f dv = 0$;

(ii) $\log f$ is a collision invariant;

(iii) f is a Maxwellian distribution, i.e. there exist $\rho \in \mathbb{R}$, $\theta > 0$ and $u \in \mathbb{R}^3$ such that $f(v) = \frac{\rho}{(2\pi\theta)^{\frac{3}{2}}} e^{-\frac{|v-u|^2}{2\theta}}$ for all $v \in \mathbb{R}^3$;

(iv) $Q(f,f) = 0$.

Exercise 2

Let $v, v_* \in \mathbb{R}^3$, and $\omega \in \mathbb{S}^2$. In the lecture we defined the post-collisional velocities (v', v'_*) associated to the pair of pre-collisional velocities (v, v_*) with the angular parameter ω as:

$$\begin{cases} v' = v - (v - v_*) \cdot \omega\, \omega, \\ v'_* = v_* + (v - v_*) \cdot \omega\, \omega. \end{cases} \tag{1}$$

We denote as $(v', v'_*)(\omega)$ the pair of post-collisional velocities defined by (1). In the literature, one may find another parametrization for the post-collisional velocities, called the *σ-representation*, defined for any $\sigma \in \mathbb{S}^2$ as

$$\begin{cases} v'' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\,\sigma, \\ v''_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\,\sigma. \end{cases} \tag{2}$$

We denote as $(v'', v''_*)(\sigma)$ the pair of post-collisional velocities defined by (2).

(i) Prove that the two parametrizations are equivalent, i.e. that for any $\omega \in \mathbb{S}^2$, there exists a unique parameter $\sigma \in \mathbb{S}^2$ such that $(v', v'_*)(\omega) = (v'', v''_*)(\sigma)$.

Prove also that for any $\sigma \in \mathbb{S}^2$ there exists a parameter ω such that $(v', v'_*)(\omega) = (v'', v''_*)(\sigma)$. Is this choice of ω unique? If not, how many possibilities are there for ω for any given σ?

(ii) Represent on a picture, for a given pair of pre-collisional velocities $(v, v_*) \in \mathbb{R}^6$, $v \neq v_*$, and a given angular parameter $\omega \in \mathbb{S}^2$, the associated pair of post-collisional velocities $(v', v'_*)(\omega)$. Represent also the vector σ associated to ω.

(iii) We have seen in the lecture that the collision kernel for the hard sphere model is given by $|(v - v_*) \cdot \omega|$, that is the collision term of the Boltzmann equation writes:

$$Q(f, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v - v_*) \cdot \omega| \left(f' f'_* - f f_* \right) d\omega \, dv_*. \tag{3}$$

Prove that in the σ-representation the hard sphere collision kernel is given by $|v - v_*|$, i.e.:

$$Q(f, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \frac{|v - v_*|}{2} \left(f'' f''_* - f f_* \right) d\sigma \, dv_*, \tag{4}$$

(where $f'' = f(v'')$ and $f''_* = f(v''_*)$).

Exercise 3

In this exercise we will study the explicit kernel of a power law potential.

In order to do so, we first introduce some basic properties of motion of a particle in $\mathbb{R}^3$. Let $U : \mathbb{R}_+ \to [0, +\infty)$ a radial potential, and the force $F : \mathbb{R}^3 \to \mathbb{R}$ associated to it defined as

$$F(x) := -\nabla_x \left(U(|x|) \right). \tag{5}$$

A particle submitted to F satisfies Newton's equation, in the sense that its position and velocity $(x(t), v(t))$ solve

$$\begin{cases} \partial_t x(t) = v(t), \\ \partial_t v(t) = F(x(t)). \end{cases} \tag{6}$$

Once fixed the initial condition $(x(0), v(0)) = (x_0, v_0)$ we know the solution to (6) is unique.

(i) Prove that the *angular momentum*[1] $L(t) := x(t) \wedge v(t)$ is conserved. Prove that the movement of the particle lies in a plane.

Hint: For two generic vectors $u, \ w \in \mathbb{R}^3$ what geometical property do $u, \ w$ and $u \wedge w$ fulfill?

(ii) Let $\mathcal{E}_c(t)$ and $\mathcal{E}_p(t)$ be respectively the *kinetic and potential energy of the particle* at time t, i.e.

$$\mathcal{E}_c(t) = \frac{1}{2} |v(t)|^2, \qquad \mathcal{E}_p(t) = U(|x(t)|). \tag{8}$$

Show that the total energy of the system $\mathcal{E}_{tot}(t) = \mathcal{E}_c(t) + \mathcal{E}_p(t)$ is conserved in time if $(x(t), v(t))$ is a solution of (6).

[1] Recall that given two vectors $u, w \in \mathbb{R}^3$ with $u \wedge w$ we denote the **vector product** between u and w, which is defined as

$$u \wedge w = \begin{pmatrix} u_2 w_3 - u_3 w_2 \\ u_3 w_1 - u_1 w_3 \\ u_1 w_2 - u_2 w_1 \end{pmatrix}. \tag{7}$$

(iii) From point (i) the motion of the particle lays in the plain spanned by x_0 and v_0. Consider the system of coordinates so that the component along the third component is zero. Furthermore on the plain of motion consider polar coordinates, so that any vector x can be represented as $x = (\rho\cos\alpha, \rho\sin\alpha, 0)$ in a suitable basis. Let $\rho(t)$, $\alpha(t)$ the polar coordinates associated to $x(t)$ (i.e. $x(t) = (\rho(t)\cos\alpha(t), \rho(t)\sin\alpha(t), 0)$). Find the expression of $\mathcal{E}_c(t)$ and $\mathcal{E}_{tot}(t)$ in terms of $\rho(t)$, $\alpha(t)$.

Assume now that U is compactly supported, that is $U(\rho) = 0$ for $\rho > \sigma$ for some real $\sigma > 0$ and decreasing in ρ. Let us assume in addition that $|x_0| > \sigma$, $v_0 = -Ve_1$ with $V > 0$.

For small times the motion of the particle is free (as long as we are outside of the support of the potential $v(t)$ is constant); we assume that initially the particle approaches the origin with impact parameter $p \in (0, \sigma)$, where the impact parameter is defined as $p = x_0 \cdot e_2$ (i.e., the trajectory can be written for small times as $x(t) = (t - C)v_0 + pe_2$ with a suitable real constant C, see also Figure 1 below).

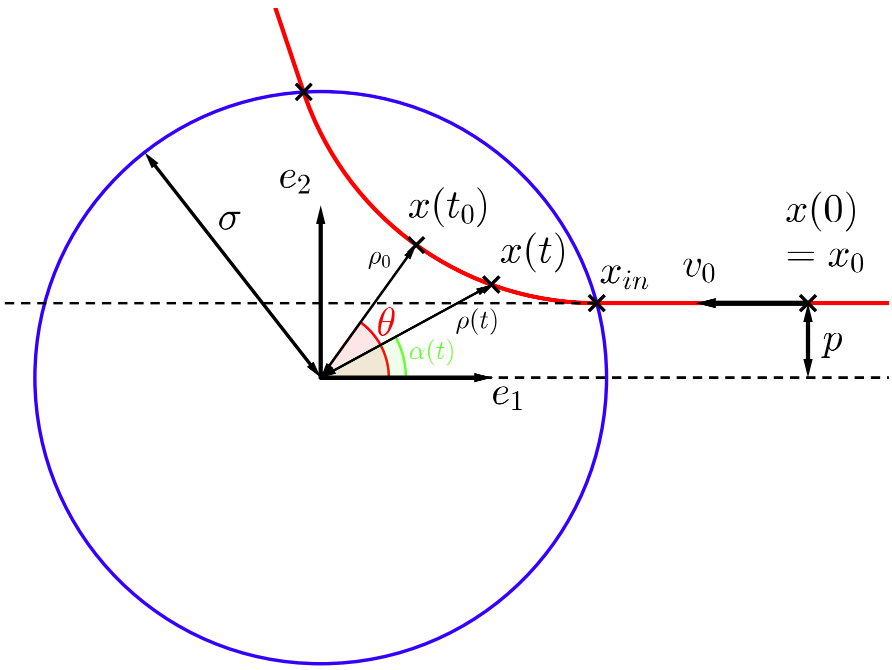

Figure 1: The movement of the particle through the support of the potential.

(iv) Using the repulsive property of the potential, prove that the distance ρ between the particle and the origin has a single minimum ρ_0.

Suppose that t_0 denotes the time at which the minimum is reached. Consider the line between the origin and $x(t_0)$ (the so-called *apse line*), and define as θ the angle between e_1 and this line. The angle θ is called the *deviation angle*.

(v) Prove that the trajectory of $x(t)$ is symmetric with respect to this minimum, i.e. we have for any $t \in \mathbb{R}$

$$\rho(t_0 + t) = \rho(t_0 - t), \qquad \alpha(t_0 + t) - \theta = -(\alpha(t_0 - t) - \theta). \tag{9}$$

(vi) In the case of the potential with cut-off, prove that the conservation of the total energy and the angular momentum respectively write:

$$\begin{cases} \frac{1}{2}\left(\dot{\rho}^2 + \rho^2\dot{\alpha}^2\right) + U(\rho) = \frac{1}{2}V^2 + U(\sigma), \\ \rho^2\dot{\alpha} = pV, \end{cases} \tag{10}$$

where $\dot{\rho}$ and $\dot{\alpha}$ denote respectively the time derivatives of ρ and α.

Hint: Consider the total energy and the angular momentum at the point x_{in}, where the particle enters the support of the potential (that is, the first time that $|x(t)| = \sigma$).

(vii) We denote as t_1 the time such that $x(t_1) = x_{in} = \sigma(\cos\alpha(t_1), \sin\alpha(t_1), 0)$. Prove that

$$\theta = \int_{t_1}^{t_0} \dot{\alpha}(t)\, dt + \arcsin\left(\frac{p}{\sigma}\right). \tag{11}$$

(viii) Prove the following identity:

$$\int_{t_1}^{t_0} \dot{\alpha}(t)\, dt = \frac{pV}{\sqrt{2}} \int_{\rho_0}^{\sigma} \frac{1}{w^2\sqrt{\frac{V^2}{2}\left(1 - \frac{p^2}{w^2}\right) - U(w) + U(\sigma)}} dw. \tag{12}$$

Hint: Use the conservation laws (10) to find an expression for $\dot{\rho}$ and $\dot{\alpha}$ in terms of ρ only, write $\dot{\alpha} = \frac{\dot{\alpha}}{\dot{\rho}}\dot{\rho}$, substitute $\frac{\dot{\alpha}}{\dot{\rho}}$ with a function of ρ only, integrate in time and change variables as $\rho(t) = w$.

(ix) Find an equation satisfied by the minimal distance ρ_0. Up to assume that we can solve this equation, deduce an explicit expression of θ (the expression (12) is of course not explicit, since it relies on determining the quantity $\dot{\alpha}$).

Consider now $U(\rho) = k\rho^{1-n}$ in its support. The explicit expression of θ reads:

$$\theta = \frac{pV}{\sqrt{2}} \int_{\rho_0}^{\sigma} \frac{1}{w^2\sqrt{\frac{V^2}{2}\left(1 - \frac{p^2}{w^2}\right) - \frac{k}{w^{n-1}} + \frac{k}{\sigma^{n-1}}}} dw + \arcsin\left(\frac{p}{\sigma}\right). \tag{13}$$

(x) Prove that, thanks to a change of variables, the deviation angle θ can be written as:

$$\theta = \int_{\lambda}^{\bar{x}} \frac{1}{\sqrt{1 - x^2 - \left(\frac{x}{b}\right)^{n-1}}} dx + \arcsin\left(\frac{p}{\sigma}\right), \tag{14}$$

with

$$\lambda = \frac{p}{\sigma}\sqrt{1 + \frac{2k}{V^2\sigma^{n-1}}}, \qquad b = p\left(\frac{V^2}{2k} + \frac{k}{\sigma^{n-1}}\right)^{\frac{1}{n-1}}, \tag{15}$$

and $\bar{x}$ solving the equation $1 - \bar{x}^2 - \left(\frac{\bar{x}}{b}\right)^{n-1} = 0$.

(xi) Finally consider the limit $\sigma \to +\infty$ (which corresponds to relaxing the cut-off on the support of the potential). Recall that the collision kernel is written as

$$B(\theta, V) = V p(\theta)\, \partial_\theta p(\theta)\,. \tag{16}$$

Prove that in the case of the inverse power law potential $U(\rho) = k\rho^{1-n}$ **without cut-off**, the collision kernel has the form:

$$B(\theta, V) = V^{\gamma} b(\theta)\,, \tag{17}$$

with $\gamma = \frac{n-5}{n-1}$, and where b is seen as a function of θ through (14).